On Non-Linear operators for Geometric Deep Learning

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Abstract

We characterize the class of (non-linear) operators that are equivariant to the action of **diffeomorphisms**, in two cases:

1. the input is a function which values are scalars
2. the input is a function which values are vectors.

The set of **Diffeomorphisms** is the biggest possible set of transformations, it appears as the invariance group of shapes.

Equivariance: What and Why?

What is equivariance?

When a transformation ϕ acts on the domain of a signal f, it induces a transformation on the signal: $L_\phi f = f \circ \phi$. A Network M is *equivariant* when it respects the transformations,

$$M[L_\phi f] = L_\phi[Mf]$$

Why look for equivariant networks?

Inductive bias, reduction of network complexity, increased accuracy for tasks related to the transformations (e.g. invariance).

Example: CNNs are equivariant to *translations*.

Diffeomorphisms: transformations of shapes

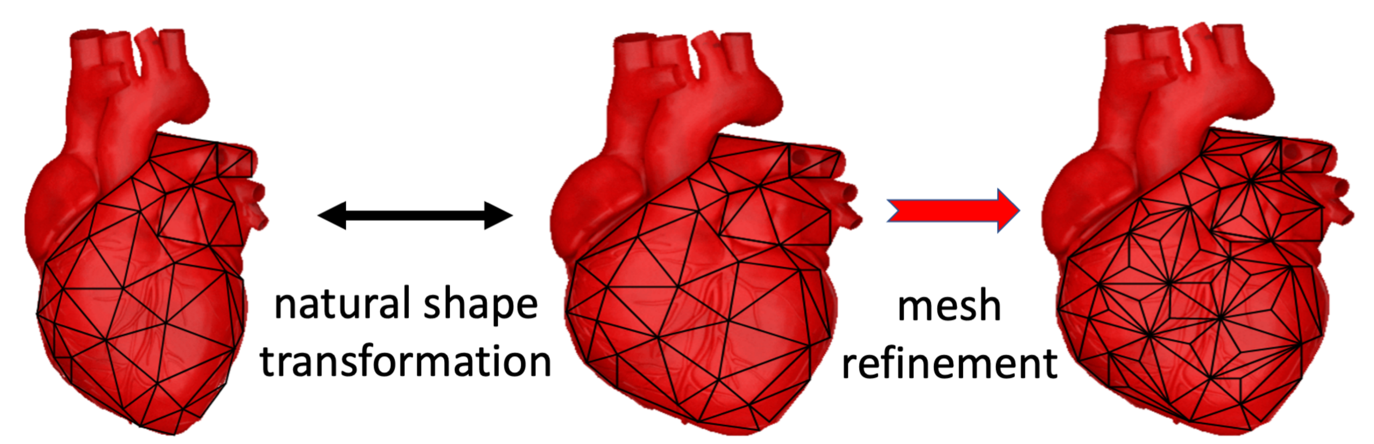
Why the group of diffeomorphisms $\mathrm{Diff}(\mathcal{M})$**?**

- Naturally appears as the symmetry group of shapes (seen as scalar valued functions)

Numerically:

- Shapes replaced by $\rightsquigarrow$ Meshed Shapes (finite dimensional).
- Transformations (symmetries) on shapes $\rightsquigarrow$ transformations on Meshes.

Example: Beating heart with triangular mesh.



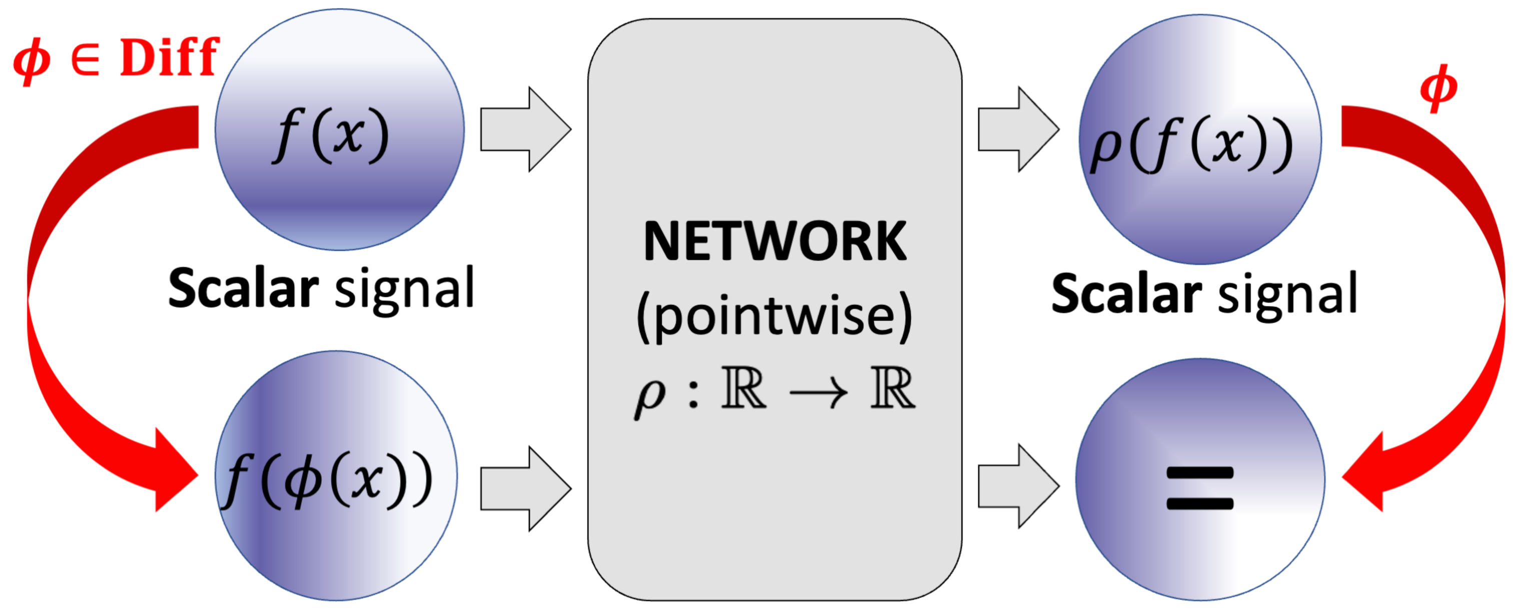
When scale of mesh $\lambda \to 0$, 'no limitation on refinement of Meshes', then recover shapes and diffeomorphisms:

Mesh $\underset{\lambda\to 0}{\to}$ Shape Symmetries of Meshes $\underset{\lambda\to 0}{\to}$ **diffeomorphisms**

Equivariant operators for shapes

Question: Can we leverage the knowledge of the symmetry group (diffeomorphisms) of shapes by designing **diffeomorphism equivariant networks**? **Can we characterize diffeomorphism-equivariant** networks for scalar valued functions?

Answer: Yes (Theorem 1) but *Very few* diffeomorphism- equivariant operators for a signal $f : \mathcal{M} \to \mathbb{R}$ that takes scalar values on its domain $\mathcal{M}$. Those operators are **point-wise non-linearities**.



Theorem 1: Equivariant operators for scalar functions

Let $\mathcal{M}$ be a connected and orientable manifold of dimension $d \geq 1$. We consider a Lipschitz continuous operator $M : L^p_\omega(\mathcal{M}, \mathbb{R}) \to L^p_\omega(\mathcal{M}, \mathbb{R})$, where $1 \leq p < \infty$. Then,

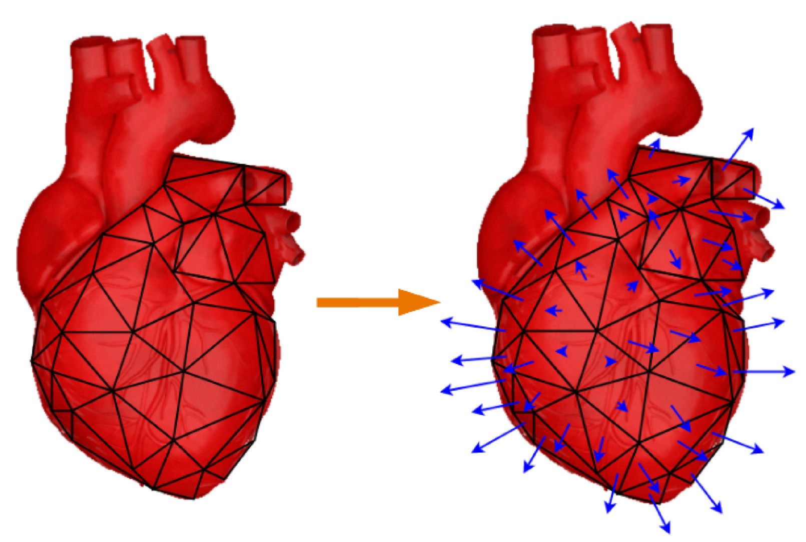
$$\forall \phi \in \mathrm{Diff}(\mathcal{M}) : \; ML_\phi = L_\phi M$$

is equivalent to the existence of a Lipschitz continuous function $\rho : \mathbb{R} \to \mathbb{R}$ that fulfills

$$\forall f \in L^p_\omega(\mathcal{M}, \mathbb{R}) \quad M[f](m) = \rho(f(m)) \quad \text{a.e.}$$

Directional shapes

- Add directional information on each point of the shape or at the center of faces of the mesh.

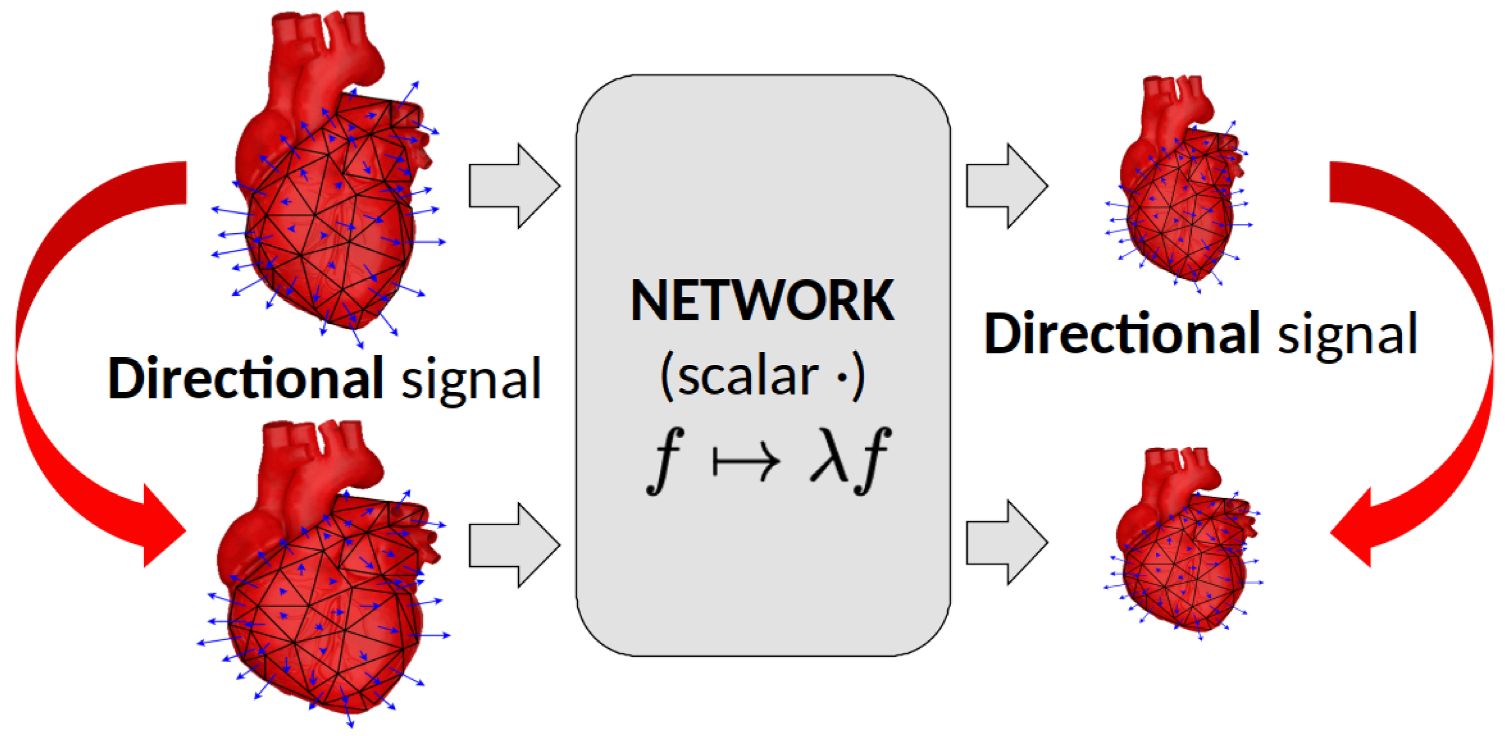


- A shape is a subset of an ambient space $\mathcal{M}$.
- The vectors that are perpendicular to the surface of the shape are in the **tangent space**, $T\mathcal{M}$, of the ambient space $\mathcal{M}$.
- **A directional shape** is (in particular) a function f :M→TM that send points of $\mathcal{M}$ to vectors in $T\mathcal{M}$.

Equivariant operators for directional shapes

Question: Can we characterize diffeomorphism-equivariant networks for vector valued functions? $f : \mathcal{M} \to T\mathcal{M}$ associates to any point of $\mathcal{M}$ a vector in the tangent space of $\mathcal{M}$.

Answer: Yes (Theorem 2) but *Even fewer* diffeomorphism-equivariant operators for a signal $f : \mathcal{M} \to T\mathcal{M}$ that takes vector values over its domain $\mathcal{M}$. Those operators are **multiplications by a scalar**.



Theorem 2: Equivariant operators for vector fields

Let $\mathcal{M}$ be a connected and orientable manifold of dimension $d \geq 1$. We consider a (Lipschitz) continuous operator $M : L^p_\omega(\mathcal{M}, T\mathcal{M}) \to L^p_\omega(\mathcal{M}, T\mathcal{M})$, where $1 \leq p < \infty$. Then,

$$\forall \phi \in \mathrm{Diff}(\mathcal{M}) : \; ML_\phi = L_\phi M$$

is equivalent to the existence of a scalar $\lambda \in \mathbb{R}$ such that

$$\forall f \in L^p_\omega(\mathcal{M}, T\mathcal{M}) : \; M[f](m) = \lambda f(m) \quad \text{a.e.}$$

References

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Paper

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